113 Class Problems: Groups and Homomorphisms

1. Let $G = \{a, b\}$ come equipped with the binary operation:

$$\begin{aligned} * : G \times G &\to G \\ (a,a) &\mapsto a \\ (a,b) &\mapsto a \\ (b,a) &\mapsto b \\ (b,b) &\mapsto b \end{aligned}$$

Is $(G, *)$ a group? Carefully justify your answer.

Solution:

$*$	a	b
a	a	a
b	b	b

$(G, *)$ is not a group because there is no identity.

$$\left.\begin{aligned} &\text{If } a = e \Rightarrow a * b = b \\ &\text{If } b = e \Rightarrow b * a = a \end{aligned}\right\} \text{Neither true}$$

2. Let $\mathbb{R}^+ = \{x \in \mathbb{R} | x > 0\}$. Prove that $(\mathbb{R}^+, \times)$ is a group. Prove that $(\mathbb{R}, +)$ and $(\mathbb{R}^+, \times)$ are isomorphic.

Solution:

First observe that $a, b > 0 \Rightarrow ab > 0 \Rightarrow \times$ is binary operation on $\mathbb{R}^+$

- Given $a, b, c \in \mathbb{R}^+$, $(ab)c = a(bc)$
- $1 \in \mathbb{R}^+$ and $1 \cdot a = a \cdot 1 = a \quad \forall a \in \mathbb{R}^+$
- Given $a \in \mathbb{R}^+$, $a^{-1} \in \mathbb{R}^+$ and $a^{-1} \cdot a = a \cdot a^{-1} = 1$

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R}^+ \\ x &\longmapsto e^x \end{aligned}$$

- Calculus $\Rightarrow f$ bijective
- $f(x+y) = e^{(x+y)} = e^x \cdot e^y = f(x) \cdot f(y) \Rightarrow f$ homomorphism

3. Let G be a group and $y \in G$. Prove that the map

$$\begin{aligned} \phi : G &\rightarrow G \\ x &\mapsto y^{-1} * x * y \end{aligned}$$

is an isomorphism. An isomorphism from a group to itself is an **automorphism**.

Solution:

- Let $\begin{aligned} \psi : G &\longrightarrow G \\ x &\longrightarrow y * x * y^{-1} \end{aligned} \Rightarrow \begin{aligned} \psi \circ \phi &= Id_G \\ \phi \circ \psi &= Id_G \end{aligned} \Rightarrow \phi$ bijection

- Let $a, b \in G$

$$\phi(a * b) = y^{-1} * a * b * y = (y^{-1} * a * y) * (y^{-1} * b * y) = \phi(a) * \phi(b)$$

$\Rightarrow \phi$ homomorphism

4. Let G be a group and $Aut(G)$ be the set of all automorphisms of G. Observe that the composition of two automorphisms is again an automorphism. Prove that composition of functions makes $Aut(G)$ a group. Hint: the hard part is showing the inverse map of an automorphism is again an automorphism.

Solution:

composition of functions is associative

- Given $f, g, h \in Aut(G)$, $(f \circ g) \circ h = f \circ (g \circ h)$
- $Id_G \in Aut(G)$ and $Id_G \circ f = f \circ Id_G = f \quad \forall f \in Aut(G)$
- Let $f \in Aut(G) \Rightarrow f$ bijection $\Rightarrow \exists\, f^{-1} : G \rightarrow G$ such that

$f \circ f^{-1} = f^{-1} \circ f = Id_G$ and f^{-1} bijective.

<u>Claim</u> f homomorphism $\Rightarrow f^{-1}$ homomorphism.

must exist as f bijective

<u>Proof</u> Let $x, y \in G$ and $x', y' \in G$ such that $f(x') = x$, $f(y') = y$

$(f^{-1}(x) = x', f^{-1}(y) = y')$

$$\Rightarrow f^{-1}(x * y) = f^{-1}(f(x') * f(y')) = f^{-1}(f(x' * y'))$$

$$= x' * y' = f^{-1}(x) * f^{-1}(y)$$

$\square$

$\Rightarrow f^{-1} \in Aut(G)$